MA 441, Fall 2010

Assignment 3.

This homework is due *Thursday* 10/14/2010.

There are total 43 points in this assignment. 36 points is considered 100%. If you go over 36 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

This assignment covers sections 3.3–3.5. Main notions you need to be familiar with co complete this assignment include (but are not limited to) *Monotone Convergence Theorem, subsequences, Cauchy sequences.*

(1) [4pt] (Exercise 3.3.2) Let $x_1 > 1$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone, hence convergent. Find the limit.

(2) [3pt] Find a mistake in the following argument:
"Let (x_n) be a sequence given by $x_1 = 1$, $x_{n+1} = 1 - x_n$. In other words, $(x_n) = (1, 0, 1, 0, 1, 0, \ldots)$. Show that $\lim(x_n) = 0.5$. Indeed, let $\lim(x_n) = x$. Apply limit to both sides of equality $x_{n+1} = 1 - x_n$:

$$\lim(x_{n+1}) = \lim(1 - x_n)$$
$$\lim(x_{n+1}) = 1 - \lim(x_n)$$
$$x = 1 - x,$$

so $x = 0.5$"

(3) [4pt] (Exercise 3.3.11) Establish convergence or divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

(4) (a) [3pt] (Exercise 3.3.12) Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $n \in \mathbb{N}$. Prove that that (x_n) converges. (*Hint*: for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k} - \frac{1}{k-1}$.)

(b) [3pt] Let K be a natural number $K \geq 2$. Let $y_n = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \cdots + \frac{1}{n^K}$, $n \in \mathbb{N}$. Prove that that (y_n) converges. (*Hint*: compare y_n to x_n.)

— *see next page* —

(5) (a) [2pt] Let $n \in \mathbb{N}$. Show that
$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \geq \frac{1}{2}.$$
(*Hint*: How many terms are there in this sum? What is the smallest term?)

(b) [3pt] (Subsection 3.5.6c) Prove that sequence
$$x_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \qquad n \in \mathbb{N}$$
diverges. (*Hint*: For $m > n$, $|x_m - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$. Put $m = 2n$ and use Cauchy criterion.)

(c) [2pt] Find a mistake if the following argument:
"Show that sequence $x_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \in \mathbb{N}$ converges. Indeed,
$$\lim \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \right) =$$
$$\lim \frac{1}{1} + \lim \frac{1}{2} + \lim \frac{1}{3} + \cdots + \lim \frac{1}{n-1} + \lim \frac{1}{n} =$$
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + 0 + 0$$
Therefore, (x_n) converges."

(6) [3pt] (Exercise 3.4.14) Let (x_n) be a bounded sequence and let
$$s = \sup\{x_n : n \in \mathbb{N}\}.$$
Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to x.

(7) (a) [3pt] (Exercise 3.4.9) Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.

(b) [3pt] Suppose that every subsequence of $X = (x_n)$ has a converging subsequence. Is it true that in this case, X must converge?

(8) (a) [3pt] (Exercise 3.5.4) Show directly from definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.

(b) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$. Is it true that if for any $\varepsilon > 0$, there is a natural number $H = H(\varepsilon)$ such that $|x_H - x_{H+1}| < \varepsilon$, then X is a Cauchy sequence? (*Hint*: look at item 5b in this assignment or at exercise 3.5.5 in Textbook).

(c) [4pt] (Exercise 3.5.9) Let $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$. Show that (x_n) is a Cauchy sequence.